Sec 6.1 Eigenvalues / eigenvectors

★ Watch 3Blue1Brown video "Eigenvectors"

By Example : $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ what are eigenvalues/vectors ?

To find eigenvalues λ, find roots of the characteristic polynomial/equation $\overbrace{\underbrace{\det(\underline{A} - \lambda \underline{I})}_{\text{polynomial}} = 0}^{\text{equation}}$

why? $(\underline{A} - 3\underline{I})\underline{v} = \underline{0} \iff \underline{A}\underline{v} = 3\underline{v}$

Example : characteristic poly/eq : $|\underline{A} - \lambda \underline{I}|$

$$= \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$
$$= \lambda^2 - 5\lambda + 4 - 4 = 0$$
$$\lambda(\lambda - 5) = 0$$

roots are $\lambda = 0$ (mult. 1)
$\lambda = 5$ (mult. 1) $\implies$ eigenvalues are $\lambda = 0$ (mult. 1), $\lambda = 5$ (mult 1)

$\underline{\lambda = 0}$: (or E_λ)

DEF "$\mathcal{E}_\lambda$" is the set of all "λ-eigenvectors" (for $\underline{A}$)

↳ $\mathcal{E}_\lambda$ is set of solutions to $\underbrace{(\underline{A} - \lambda \underline{I})}_{\text{matrix}} \underbrace{\underline{v}}_{\text{vector}} = \underline{0}$

$\mathcal{E}_0$ = solutions to $(\underline{A} - 0\underline{I})\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

augmented mat. $\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc:c} \boxed{1} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ ← $a + 2b = 0$
$a = -2b$

b free

$\mathcal{E}_0$ is all $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = b\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (b free $\in \mathbb{R}$)

So $\mathcal{E}_0 = \text{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

★ $\underline{0}$ is never called an eigenvector

pick $\underline{v_1} = \begin{bmatrix}-2\\1\end{bmatrix}$ as <u>an</u> eigenvector.

Now find $\mathcal{E}_5$:

solve $(\underline{A} - 5I)\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \longleftarrow \begin{array}{l} 2a - b = 0 \\ a = \frac{1}{2}b \end{array}$$

b free

$$\mathcal{E}_5 = \left\{\text{all } \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\frac{1}{2}b\\b\end{bmatrix} = b\begin{bmatrix}1/2\\1\end{bmatrix},\ b \in \mathbb{R} \text{ free}\right\}$$

$$= \text{span}\left\{\begin{bmatrix}1/2\\1\end{bmatrix}\right\} \quad \left(\text{alt} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}\right)$$

$\underline{v_2} = \begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector for $\underline{A}$.

★ Fact: $n \times n$ matrix can have <u>at most</u> n <u>lin</u> <u>indep</u>. eigenvectors.

<u>Ex</u> $\underline{A} = \begin{bmatrix}7 & 1\\ 0 & 7\end{bmatrix}$ e-values: $|\underline{A} - \lambda I| = \begin{vmatrix}7-\lambda & 1\\ 0 & 7-\lambda\end{vmatrix} = (7-\lambda)^2$

$$\longrightarrow (7-\lambda)^2 = 0$$

so eigenvalue(s) is $\lambda = 7$ (mult. 2)

<u>$\lambda = 7$</u> Find solutions to $(\underline{A} - 7\underline{I})\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\left(\text{null}(\underline{A} - 7\underline{I})\right)$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc:c}0 & \boxed{1} & 0\\ 0 & 0 & 0\end{array}\right] \longleftarrow \begin{array}{l} b = 0 \\ a \text{ free.} \end{array}$$

a free pivot

$\mathcal{E}_7 = \text{all } \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix}, (a \text{ free})$

so $\underline{v} = \begin{bmatrix}1\\0\end{bmatrix}$ is e-vect.

$\mathcal{E}_7 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

Notice: $$\underset{1}{\underset{\parallel}{\dim(\mathcal{E}_7)}} \le \underset{2}{\underset{\parallel}{\left(\text{multiplicity of } \lambda = 7\right)}}$$

<u>Def</u>: $\dim(\mathcal{E}_\lambda) =$ "<u>geometric multiplicity</u>"

<u>Def</u>. ("multiplicity of root λ") = "<u>algebraic multiplicity</u>"

$$\boxed{\underline{\text{Thm}}: \quad \dim(\mathcal{E}_\lambda) \le \left(\text{mult. of } \lambda \text{ as a root}\right)}$$